Interest Meeting

DV Math Club — Leader's Edition

September 9, 2026

Statements, sources, clarifying questions, hints, and full solutions.

How to run these. State the problem. Answer clarifying questions only, and answer them exactly as written below, because several of these problems are ruined by a loose answer to an innocent question. Give the hint after ten minutes. Hold the solution until at least one group has a wrong answer they are willing to defend.

On the wording. The statements are reproduced exactly as they appear on the student sheet. All six problems are folklore: each has a first appearance in print, but none has a single authoritative text, and the wording drifts from source to source. Each problem carries a provenance note recording the earliest source that could be traced.

Problem 1. The Twenty-One Game

Imagine a pile of twenty-one stones. Two players, A and B, alternate moves. On your turn, you can choose to remove either one, two, or three stones from the pile. The player who takes the last stone wins.

Which player can force a win, and what is the winning strategy?

Provenance. Folklore. It is the smallest interesting case of a subtraction game, and a descendant of Bachet's "game of 100" from *Problèmes plaisants et délectables* (1612), in which players alternately add a number from 1 to 10 to a running total and the first to reach 100 wins. No canonical wording exists for the 21-stone form; the statement above is the standard one.

Clarifying questions

- *Must I take at least one stone?* Yes. Passing is not allowed.
- *Can I take three when only two remain?* No. You may never take more stones than are in the pile.
- *Does the player who takes the last stone win or lose?* Wins. This is normal play. The other convention changes the answer, and is problem 1′ below.

Hint

Work backwards. Which pile sizes would you hate to be handed?

Solution

The first player wins. Take one stone, leaving twenty. Thereafter, always leave a multiple of four.

Call a position *losing* if the player about to move will lose against correct play. Zero stones is losing, since that player has already lost.

- From a pile that is a multiple of four, every legal move (remove 1, 2, or 3) leaves a pile that is *not* a multiple of four.
- From a pile that is not a multiple of four, say $4q + r$ with $r \in \{1, 2, 3\}$, removing exactly r stones leaves $4q$, a multiple of four.

So a player facing a multiple of four must hand over a non-multiple, and a player facing a non-multiple can always hand back a multiple. The pile strictly decreases each turn, so the game ends, and it ends at 0, which is a multiple of four. Whoever first faces a multiple of four is therefore the loser.

Since $21 = 4 \cdot 5 + 1$, the first player removes one stone and wins.

Extensions

1′. **Misère play.** Same game, but the player who takes the last stone *loses*. Now the losing positions are the piles congruent to 1 modulo 4. Since $21 \equiv 1 \pmod 4$, the *second* player wins. This reversal is worth showing, because students assume the strategy transfers.

1′. **General.** Remove between 1 and m stones: leave multiples of $m + 1$.

1′. **Bachet's original.** Alternately add 1 to 10 to a running total; first to reach 100 wins. Reach 89, and before that 78, 67, and so on down to 1. The first player wins by naming 1.

Problem 2. Coins on a Table

There exists some table with indeterminate size such that it is large enough to fit at least one coin. Two players alternate turns placing identical circular coins on this circular table. A coin may not overlap a coin already placed, and it may not hang over the edge of the table. Once placed, a coin cannot be moved. The player who cannot place a coin loses.

Which player can force a win, and how?

Provenance. Folklore, long-standing in the recreational mathematics literature and with no traceable first author. The statement above is the standard one. The essential hypothesis is that the table is symmetric about a centre point and that at least one coin fits on it.

Clarifying questions

- *Are the coins all the same size?* Yes, identical.
- *Can coins touch?* Yes. They may touch; they may not overlap.
- *Is the table large enough for at least one coin?* Yes.
- *Do we know how many coins fit?* No, and you will not need to.
- *Does the table have to be circular?* The proof below needs only that the table looks the same after rotation by 180° about some point. A square or a rectangle also works. A triangle does not.

Hint

The table has a centre. Use it once, and then never again.

Solution

The first player wins.

1. First move: place a coin with its centre exactly at the centre of the table.
2. Every later move: whenever the opponent places a coin, place your coin at the point diametrically opposite through the centre of the table, that is, at the image of their coin under rotation by 180° about the centre.

The mirror move is always legal. Write σ for the rotation by 180° about the centre of the table.

- It lies on the table, because σ maps the table onto itself.
- It does not overlap the central coin. The central coin is fixed by σ, and the opponent's coin did not overlap it, so its image does not either.
- It does not overlap any earlier coin. After each of the first player's replies, the set of coins on the table is symmetric under σ. The opponent's new coin overlapped none of those, so applying σ to the whole picture shows that its image overlaps none of them either.

So the first player always has a reply available. Each coin covers a fixed positive area of a table of finite area, so the game ends after finitely many moves. The player who runs out of moves is therefore the second player.

Discussion to raise afterwards

The argument proves who wins without ever determining how many coins fit on the table, which is a genuinely hard packing question. Ask the room where the proof breaks for a triangular table. The answer is that the first move has no distinguished fixed point to occupy, since a triangle has no centre of 180° symmetry.

Problem 3. The Mutilated Chessboard

A standard 8×8 chessboard has two diagonally opposite corner squares removed, leaving 62 squares. Each domino covers exactly two squares that share an edge.

Can 31 dominoes be placed so as to cover the remaining 62 squares exactly, with no overlaps and no gaps?

Provenance. Posed by the philosopher Max Black in *Critical Thinking* (Prentice-Hall, 1946), and popularised by Martin Gardner in *Scientific American*. The formulation above is the standard modern one and matches Black's problem. Black's own sentences are under copyright and are not reproduced.

Clarifying questions

- *Which two corners?* Two that are diagonally opposite, for example a1 and h8.
- *May dominoes be placed diagonally?* No. A domino covers two squares sharing an edge, so it is horizontal or vertical.
- *May a domino hang off the board or overlap another?* No.
- *Do I have to use all 31?* Yes; 31 dominoes cover 62 squares exactly, so a full cover uses all of them.

Hint

Count something that no legal placement can change.

Solution

No such covering exists.

Colour the board in the usual alternating fashion. Any two squares that share an edge have different colours, so *every domino covers exactly one black square and one white square*. Hence 31 dominoes cover 31 black squares and 31 white squares.

Two diagonally opposite corners always have the same colour. Removing them leaves 32 squares of one colour and 30 of the other. Since $32 \neq 31$, no covering exists. □

Why this problem is on the sheet

A search over placements is hopeless; the number of arrangements is astronomical. The colour

count is a quantity preserved by every legal move, so it rules out the entire infinite family of attempts in one line. That is the method of invariants, and it is the transferable content here.

The follow-up, which is the better half

Remove any *one* black square and any *one* white square, from anywhere on the board. The counting objection is now gone. Does a covering always exist?

Yes, always. Draw a closed tour of all 64 squares that steps only between edge-neighbours: run up column 1, down column 2, up column 3, and so on, then return along the top row to the start. This is a single cycle through all 64 squares, and it alternates in colour. Deleting two squares of opposite colour cuts the cycle into two arcs, each of even length, and an even arc is covered by laying dominoes along it end to end. □

This converse is due to Ralph Gomory (1973), and the tour argument is the standard proof of it.

Problem 4. Guess My Polynomial

I am thinking of a polynomial $f(x)$. Every one of its coefficients is a non-negative integer. You get two turns. Each turn, you can name any number and I will tell you the value of $f(x)$.

Can you always determine my polynomial exactly?

Provenance. Modern folklore, widely circulated in Putnam training sets and puzzle collections; no original author is traceable. The formulation above is the standard one, in which the second query may depend on the answer to the first.

Clarifying questions

- *Are the coefficients integers, or just non-negative?* Both: non-negative integers. The problem is false without either condition.
- *May my second question depend on the first answer?* Yes, and it must. This is the whole problem.
- *May I name any number, or only integers?* Any number you like; the solution below uses only positive integers.
- *Is there a bound on the degree?* No.
- *Could the polynomial be constant, or zero?* Yes, both are allowed.

Hint

Your first question should bound how large a coefficient can possibly be. Your second question should then read the coefficients off in a suitable base.

Solution

Ask for $p(1)$. Then ask for $p(p(1)+1)$.

Write $p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_dx^d$ with every c_i a non-negative integer.

First question. The answer is

$$S \;=\; p(1) \;=\; c_0 + c_1 + \cdots + c_d.$$

Because the coefficients are non-negative, no single coefficient can exceed the sum, so $0 \le c_i \le S < S+1$ for every i.

Second question. Ask for $p(S+1)$. The answer is

$$p(S+1) \;=\; c_0 + c_1(S+1) + c_2(S+1)^2 + \cdots + c_d(S+1)^d,$$

and every digit c_i satisfies $0 \le c_i \le S$. That is precisely the base-$(S+1)$ representation of the number $p(S+1)$. Base representations are unique, so writing $p(S+1)$ in base $S+1$ recovers the coefficients, in order, and determines p completely. □

Worked example

Suppose the answers are $p(1) = 6$ and $p(7) = 708$. Then

$$708 \;=\; 2 \cdot 343 \;+\; 0 \cdot 49 \;+\; 3 \cdot 7 \;+\; 1 \;=\; (2\,0\,3\,1)_7,$$

so $p(x) = 2x^3 + 3x + 1$. As a check, the coefficients sum to $6 = p(1)$.

Two edge cases to have ready

- If $p(1) = 0$, then all coefficients are non-negative and sum to zero, so p is identically zero. You are already finished and the second question is unnecessary.
- If $p(1) = 1$, base 2 is used, and the coefficients are all 0 or 1. The argument goes through unchanged.

Where the hypotheses are load-bearing

Drop *integer* and the base-$(S+1)$ reading is meaningless. Drop *non-negative* and $p(1)$ no longer bounds the coefficients: for instance $x^2 - x$ and the zero polynomial agree at $x = 1$, and the trick has nothing to work with. Ask the room to break the problem themselves; it is more instructive than the solution.

Problem 5. Twelve Coins

You have 12 coins. They all appear identical but you happen to know that one is counterfeit and fake with a different weight from the other 11 whose weights are identical. You have a balance scale, which you may use three times, and you want to determine which coin is counterfeit. How do you do this?

Provenance. The earliest traced appearance of a balance puzzle of this type is E. D. Schell, "Weighed and found wanting", *American Mathematical Monthly* **52** (1945), p. 42, which uses eight coins. The twelve-coin version appeared shortly after, submitted by Howard D. Grossman, *Scripta Mathematica* (December 1945). Freeman Dyson gave a general solution in "The problem of the pennies", *The Mathematical Gazette* **30** (1946), pp. 231–234. Those texts are under copyright; the statement above is the standard modern formulation of Grossman's version.

Clarifying questions

- *Do I have to say whether it is heavy or light, or only which coin?* Only which coin. Groups that also pin down heavy or light have solved the harder classical version, and the solution below does exactly that, so accept either.
- *Is exactly one coin counterfeit?* Yes, exactly one.
- *Does the scale give a number?* No. It reports only left heavier, right heavier, or balanced.
- *Must the pans hold equal numbers of coins?* Yes, otherwise the result carries no information. Coins may be left off the scale entirely, and doing so is essential.
- *May a later weighing depend on earlier results?* Yes.
- *Do I have any coin known to be genuine at the start?* No.

Hint

Count the outcomes before you weigh anything. Three weighings give $3 \times 3 \times 3 = 27$ distinguishable results. There are 12 possible answers to the question as asked, and $12 \times 2 = 24$ if you also want heavy or light, so almost no information may be wasted. In particular the second weighing cannot simply split a pan in half: it must move coins from one pan to the other, and set some aside.

Solution

Number the coins 1 through 12.

Weighing 1. $\{1, 2, 3, 4\}$ against $\{5, 6, 7, 8\}$.

Case A: they balance. The fake is among $9, 10, 11, 12$, and coins 1 through 8 are known genuine.

- **Weighing 2:** $\{9, 10, 11\}$ against $\{1, 2, 3\}$.
- *Balanced:* the fake is 12. **Weighing 3:** 12 against 1, which reveals heavy or light.
- *Left heavy:* one of $9, 10, 11$ is heavy. **Weighing 3:** 9 against 10. The heavier is the fake; if they balance, 11 is heavy.
- *Left light:* one of $9, 10, 11$ is light. **Weighing 3:** 9 against 10. The lighter is the fake; if they balance, 11 is light.

Case B: $\{1, 2, 3, 4\}$ is heavy. So either one of $1, 2, 3, 4$ is heavy, or one of $5, 6, 7, 8$ is light. Coins 9 through 12 are genuine.

- **Weighing 2:** $\{1, 2, 5\}$ against $\{3, 4, 6\}$.
- *Balanced:* the suspects $1, 2, 3, 4, 5, 6$ are cleared, so the fake is 7 or 8, and it is light. **Weighing 3:** 7 against 8; the lighter one is the fake.
- *Left heavy:* either 1 or 2 is heavy, or 6 is light. **Weighing 3:** 1 against 2. The heavier is the fake; if they balance, 6 is light.
- *Left light:* either 3 or 4 is heavy, or 5 is light. **Weighing 3:** 3 against 4. The heavier is the fake; if they balance, 5 is light.

Case C: $\{1, 2, 3, 4\}$ is light. Identical to Case B with the words *heavy* and *light* interchanged throughout.

Why the second weighing looks so strange

After weighing 1, Case B leaves eight live hypotheses: $1H, 2H, 3H, 4H, 5L, 6L, 7L, 8L$. Two further weighings distinguish at most 9 outcomes, so each of the three results of weighing 2 must leave at most three hypotheses. Putting $\{1, 2\}$ with 5 on one side and $\{3, 4\}$ with 6 on the other, and setting 7 and 8 aside, splits the eight hypotheses as $3 + 2 + 3$. Any simpler-looking weighing splits them worse and fails.

Extension

With w weighings, the counting bound is $(3^w - 3)/2$ coins, which gives 12 for $w = 3$ and 39 for $w = 4$, and both bounds are attainable. Ask the room to derive the bound before you tell them the value.

Problem 6. The Josephus Problem

There are n people standing in a circle, numbered from 1 to n. Starting from person 1, counting proceeds in clockwise direction. In each step, exactly $k-1$ people are skipped, and the k-th person is eliminated from the circle. The counting then resumes from the next person, and the process continues until only one person remains. Take $k = 2$.

(a) Which number survives when $n = 41$?

(b) Which number survives for a general n? Find a rule that lets you answer instantly for any n, including $n = 100$.

Provenance and a correction. Two different problems travel under this name, and an earlier draft of this sheet conflated them.

The historical story. Flavius Josephus, *The Jewish War*, Book III, describes forty-one men trapped in a cave at Jotapata who agree to die in turn rather than surrender. In the version that reached recreational mathematics, **every third man** is killed, and Josephus survives by standing in the right place. That version has **two** survivors, at positions 31 and 16.

The mathematical standard. The problem now called *the* Josephus problem eliminates **every second** person and asks for the last survivor, as stated above. It is treated at length in Graham, Knuth and Patashnik, *Concrete Mathematics* (1989), §1.3, which is where the clean binary answer comes from. A related counting-out puzzle with fifteen and fifteen appears in Bachet (1612).

The statement above is the every-second version, which is the one with the elegant answer. The every-third version is set as an extension below.

Clarifying questions

- *What is k?* $k = 2$, as stated. So exactly one person is skipped each step.
- *Who is eliminated first?* Number 2. The count starts at number 1, who is skipped.
- *Does the count continue past eliminated numbers?* Yes. The circle closes up and the counting continues without pause from the position after the one just removed.
- *Does the second pass restart at number 1?* Not necessarily. There is no restarting; it is one continuous count around a shrinking circle.
- *Is the answer a position or a person?* A position in the original numbering.

Hint

Do it by hand for $n = 2, 3, 4, \ldots, 8$ and write the survivors in a row. Then write n in binary.

Solution

Let $J(n)$ be the surviving number.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15

The values reset to 1 at every power of two and then climb through the odd numbers. That is the whole answer:

Write $n = 2^m + L$ with $0 \le L < 2^m$. Then $J(n) = 2L + 1$.

Proof. Suppose $n = 2k$ is even. One pass around the circle removes all the even positions and leaves the k odd positions, with number 1 about to be counted again. This is the same problem with k people, and position j of the smaller circle was position $2j - 1$ of the original, so

$$J(2k) = 2J(k) - 1.$$

Suppose $n = 2k + 1$ is odd. One pass removes the even positions and then also removes number 1, leaving k numbers with number 3 about to be counted. The same relabelling gives

$$J(2k + 1) = 2J(k) + 1.$$

With $J(1) = 1$, induction on n gives $J(2^m + L) = 2L + 1$: if $n = 2^m + L$ is even then L is even and $J(n) = 2J(2^{m-1} + L/2) - 1 = 2(L + 1) - 1 = 2L + 1$, and the odd case is identical. □

The binary form

The two recurrences both say: shift the binary digits of n left by one place and move the digit that falls off the top to the bottom. Iterating from $J(1) = 1$ rotates the leading 1 of n all the way round to the end. So:

Write n in binary, move the leading 1 *to the back, and read the result.*

- $n = 41 = (101001)_2 \longrightarrow (010011)_2 = 19$. Check: $41 = 32 + 9$, so $J(41) = 2 \cdot 9 + 1 = 19$.
- $n = 100 = (1100100)_2 \longrightarrow (1001001)_2 = 73$. Check: $100 = 64 + 36$, so $J(100) = 2 \cdot 36 + 1 = 73$.

Extension: the historical version

Forty-one men in a circle, **every third** man eliminated, and the process stops when two men remain. The order of elimination is

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 1, 5, 10, 14, 19, 23, 28, 32, 37, 41, 7, 13, 20, 26, 34, 40,

8, 17, 29, 38, 11, 25, 2, 22, 4, 35,

leaving positions 16 and 31. Josephus, in the story, takes one of those two places.

No formula as clean as the binary rotation is known for the every-third case. The recurrence

$$J_k(n) = \big(J_k(n - 1) + k\big) \bmod n, \qquad J_k(1) = 0$$

(with positions numbered from 0) solves it for any step size k in n steps, and is the right thing to hand a student who owns a computer.

Sources consulted: Wikipedia entries for the mutilated chessboard problem, the Josephus problem, and balance puzzles; Graham, Knuth and Patashnik, *Concrete Mathematics*, §1.3; the bibliographic notes in the puzzle literature for Schell (1945), Grossman (1945) and Dyson (1946).